NORTHERN BEACHES SECONDARY COLLEGE

MANLY SELECTIVE CAMPUS

Year 12

Trial Examination

2022

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Write your student number on the front of every booklet.
- In Questions 11 to 14 show relevant mathematical reasoning and/or calculations.
- NESA approved calculators may be used.
- Weighting:30%

Section I Multiple Choice

- 10 marks
- Attempt all questions.
- Answer Sheet provided
- Allow about 15 minutes for this section

Section II Free Response

- 60 marks
- Start a separate booklet for each question.
- Each question is of equal value.
- All necessary working should be shown in every question.
- Allow about 1 hour and 45 minutes for this section.

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10.

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1. Which of the following parametric equations represents a circle that passes through the origin?

   A. $x = 3 + 3\cos\theta,\ y = 4 + 3\sin\theta$

   B. $x = 3 + 4\cos\theta,\ y = 4 + 4\sin\theta$

   C. $x = 3 + 5\cos\theta,\ y = 4 + 5\sin\theta$

   D. $x = 3 + 7\cos\theta,\ y = 4 + 7\sin\theta$

2. The vector $\underset{\sim}{u}$ has a magnitude of 6 units and makes an angle of 135° with the horizontal.

   Which of the following is vector $\underset{\sim}{u}$?

   A $3\sqrt{2}\,\underset{\sim}{i} + 3\sqrt{2}\,\underset{\sim}{j}$

   B $-3\sqrt{2}\,\underset{\sim}{i} + 3\sqrt{2}\,\underset{\sim}{j}$

   C $2\sqrt{3}\,\underset{\sim}{i} + 2\sqrt{3}\,\underset{\sim}{j}$

   D $-2\sqrt{3}\,\underset{\sim}{i} + 2\sqrt{3}\,\underset{\sim}{j}$

3. Which one of the following is not a Bernoulli random variable?

   A. The number of P’s when a letter is chosen at random from the word PINEAPPLE.

   B. The number of vowels when a when a letter is chosen at random from the word MATHS.

   C. The number of heads when a fair coin is tossed twice.

   D. The number of 6’s when a fair die is rolled once.

4. The derivative of $y = \cos^{-1}[2f(x)]$ is :

A. $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-[f(x)]^2}}$

B. $\dfrac{dy}{dx} = -\dfrac{2}{\sqrt{1-4[f(x)]^2}}$

C. $\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-4[f(x)]^2}}$

D. $\dfrac{dy}{dx} = -\dfrac{2f'(x)}{\sqrt{1-4[f(x)]^2}}$

5. The diagram shows a trapezium OPQR.

NOT TO SCALE

If $\overrightarrow{OP} = \underset{\sim}{p}$, $\overrightarrow{OR} = \underset{\sim}{r}$ and $\overrightarrow{OP} = 3\overrightarrow{RQ}$, then $\overrightarrow{PQ}$ equals

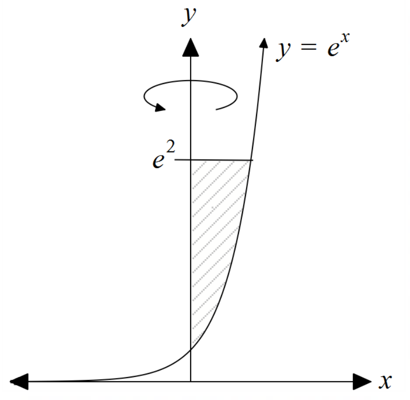
A. $\frac{4}{3}\underset{\sim}{p} + \underset{\sim}{r}$

B. $\frac{2}{3}\underset{\sim}{p} - \underset{\sim}{r}$

C. $\frac{2}{3}\underset{\sim}{p} + \underset{\sim}{r}$

D. $\underset{\sim}{r} - \frac{2}{3}\underset{\sim}{p}$

6. A solid of revolution is to be formed by rotating the area enclosed by the function $y = e^x$, the line $y = e^2$, and the y-axis about the y-axis.



Which of the following would give the volume of the solid of revolution?

A. $V = \pi \int_0^2 (\ln x)^2 \, dx$

B. $V = \pi \int_0^2 e^{2x} \, dx$

C. $V = \pi \int_1^{e^2} (\ln y)^2 \, dy$

D. $V = \pi \int_1^{e^2} e^{2y} \, dy$

7. If $\int_0^k \frac{1}{4+x^2} \, dx = \frac{\pi}{6}$, what is the value of k?

A. 1

B. $\frac{1}{2}$

C. $\sqrt{3}$

D. $2\sqrt{3}$

8. How many solutions does the equation $\sin 4x - \sin 2x = 0$ have for $0 \leq x \leq 2\pi$?

A. 7

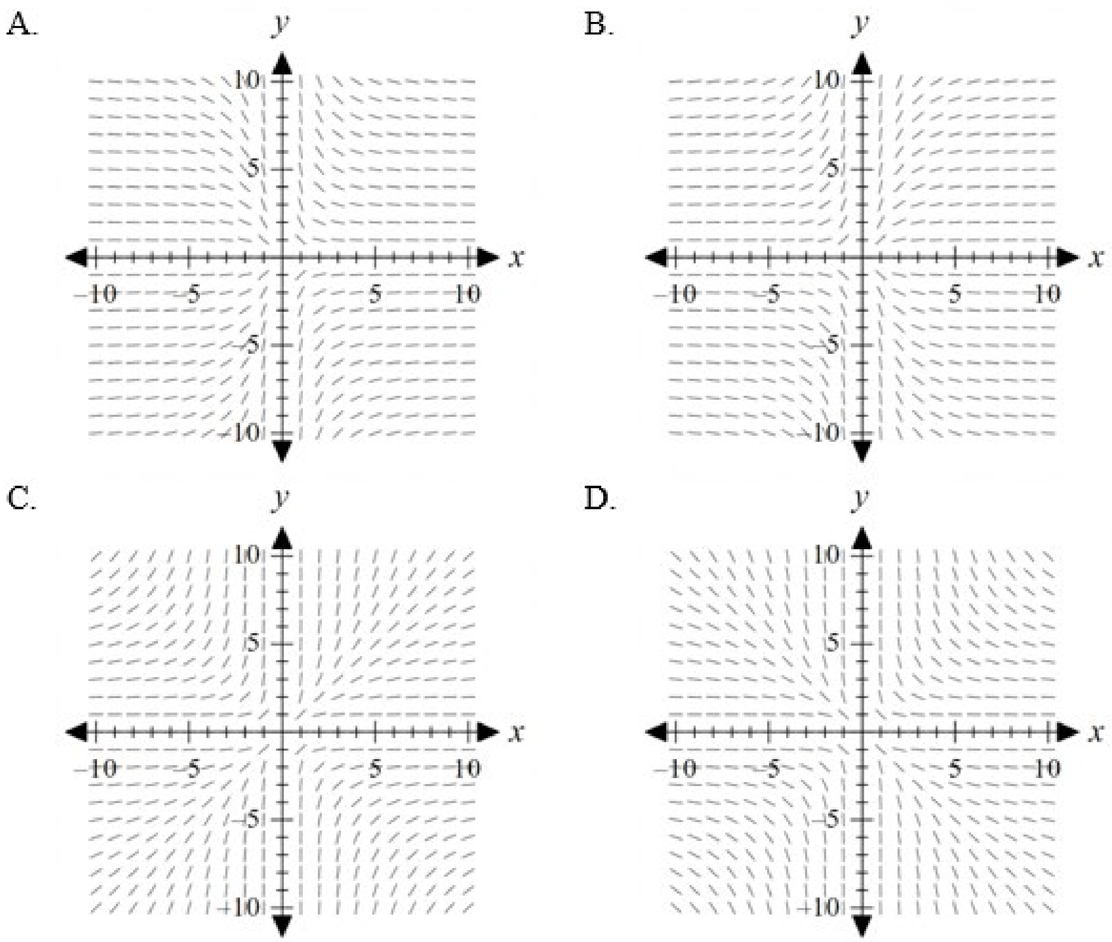
B. 8

C. 9

D. 10

9. A differential equation is given to be $\dfrac{dy}{dx} = \dfrac{y}{x^2}$

Which of the following best represents the direction field of the differential equation?



10. In NSW in 2021 approximately 70,000 students started school in kindergarten. 80% of these students could already write their name. If random samples of 20 kindergarten students were selected, what would be the shape of the sampling distribution of $\hat{p}$ - the proportion of children who can already write their name?

A. Positively skewed

B. Negatively skewed

C. Approximately normal

D. Uniform

End of Multiple Choice

Section II

60 marks - Attempt Questions 11 – 14

Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11: Use new writing book for this question **15 marks**

a) Solve $\dfrac{x}{x+1} \leq 5$ **3**

b) There are 26 different time periods in which classes at a high school can be scheduled.

If 508 classes need to be scheduled next term, what is the minimum number of different rooms needed to accommodate all classes? **1**

c) Find the term independent of x in the expansion of $\left(3x^2 - \dfrac{1}{x}\right)^{12}$ **3**

d) A polynomial *P(x)* has a remainder of $2x+1$ when divided by $x^2 - 5x + 6$.
What is the remainder when *P(x)* is divided by $x - 3$? **2**

e) Solve $\sqrt{2}\cos\left(x+\dfrac{\pi}{4}\right)+\sqrt{2}\cos\left(x-\dfrac{\pi}{4}\right)=1$ for $-\pi \leq x \leq \pi$ **3**

f) A group of 10 people arrive to eat at a restaurant. The only seating available for them is at two circular tables, one that seats six people and another that seats four. Assuming that the seating arrangement is random, what is the probability that a particular couple will be seated at the same table? (The couple do not have to sit next to each other). **3**

End of Question 11

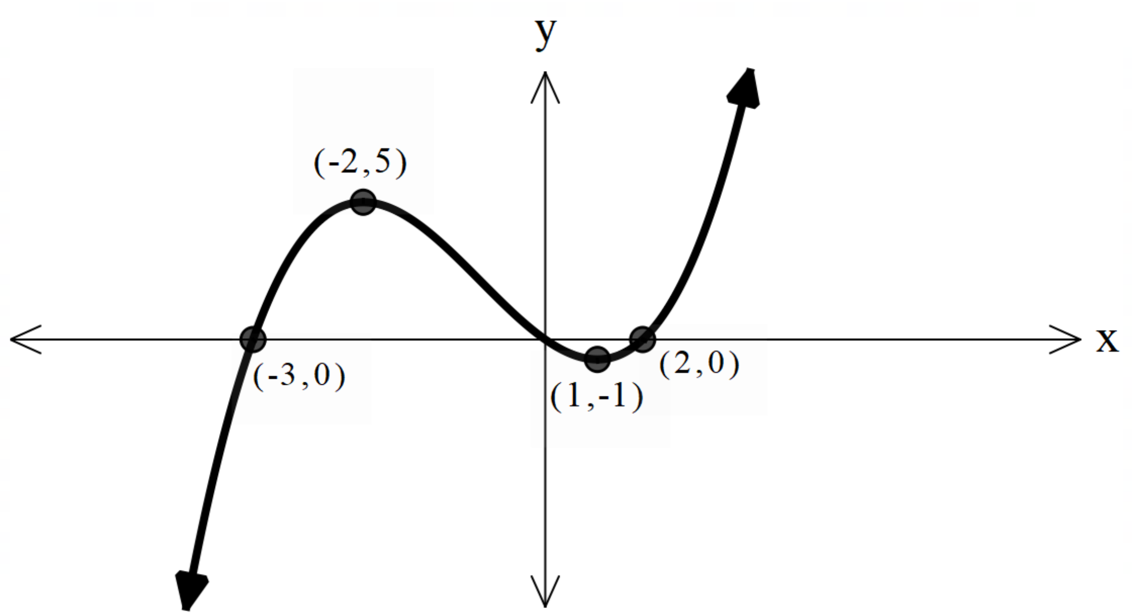
Question 12: Use new writing book for this question **15 marks**

a) Explain why the statement $2 + 4 + \ldots + 2n = n(n-1) + 2$, for $n \geq 1$ cannot be proved by mathematical induction. Support your explanation with working. **2**

b) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{4 - 4x^2}}\, dx$ **2**

c) The vectors $\underset{\sim}{p} = \begin{pmatrix} -6 \\ a \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} a+1 \\ -7 \end{pmatrix}$ are parallel. Find the possible values of a. **2**

d) The graph of $y = f(x)$ is shown.



Draw the graph of $y = |f(\,|x|\,)|$ **3**

e) The random variable X has a binomial distribution with parameters n=50 and p=0.3.

(i) Find the standard deviation. **1**

(ii) Find the probability that X is equal to the mean of the distribution. **2**

f) Solve the differential equation $\frac{dy}{dx} = y^3$ for y, given when $x = 0, y = 1$.

Give your answer in the form $y = f(x)$. **3**

End of Question 12

Question 13: Use new writing book for this question **15 marks**

a) Tomatoes are considered to be either determinate or indeterminate.

Mario buys tomatoes from his local store where the tomatoes are twice as likely to be sourced from Farm A than Farm B.

It is known that 60% of Farm A's tomatoes are determinate while 70% of Farm B's tomatoes are determinate. Mario cannot tell the difference between these tomatoes from their appearance.

(i) Show that the probability that a randomly selected tomato is determinate is 19/30. **1**

(ii) Mario buys ten tomatoes. Find the probability, correct to two decimal places, that no more than eight of these are determinate. **2**

(iii) Mario buys 33 tomatoes to make a batch of tomato paste and knows from his grandmother that at least 55% of the tomatoes should be determinate to achieve a fine paste.
By using a normal approximation to the sample proportion, determine the approximate probability that the tomato paste produced will be considered fine. **3**

Table of Standard Normal Probabilities

Table entry for *z* is the area under the standard normal curve to the left of *z*

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Question 13 continues on next page

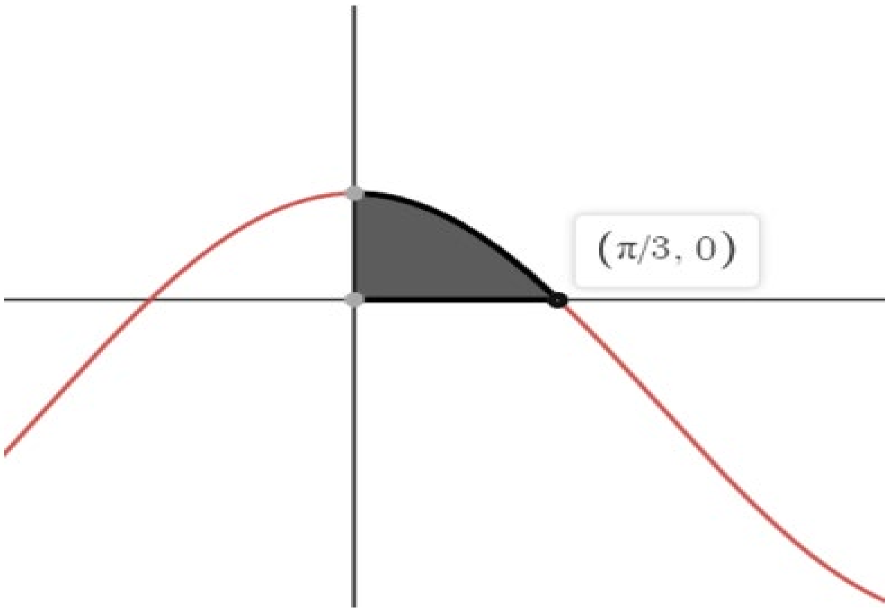
Question 13 continued

b) Solve the equation $\sin\theta + \cos\theta = \frac{1}{2}$ for $[0, 2\pi]$. Answer in radians to three decimal places. **3**

c) Suppose $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underset{\sim}{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and θ is the acute angle between them.

Show that the exact value of $\sin 2\theta$ is $\frac{4}{5}$. Give clear reasoning for your answer. **3**

d) The diagram below shows part of the curve $y = 2\cos x - 1$. The curve intersects the x-axis at the point $\left(\frac{\pi}{3}, 0\right)$. The shaded region is enclosed by the curve and the coordinate axes.



Determine the volume of the solid formed when the shaded region is rotated $360°$ about the x-axis. **3**

End of Question 13

Question 14: Use new writing book for this question **15 Marks**

a) Use mathematical induction to prove that $8^n - 5^n$ is a multiple of 3 for all positive integers n. **3**

b) Find the exact value of $\cos\left(\tan^{-1}(-3)\right)$ **2**

c) A 500 L tank contains 200 L of brine (salt in water) with 50 kg of salt dissolved.

Pure water is pumped into the tank at 20 L/min. At the same time, the perfectly mixed brine in the tank is pumped out of the tank at 15 L/min.

(i) Explain why the amount of salt m kg in the tank after t minutes can be modelled by the differential equation $\frac{dm}{dt} = -\frac{3m}{40+t}$. **1**

(ii) Hence, find m as a function of t. **3**

(iii) How many kilograms of salt is in the tank when it begins to overfill? Give your answer correct to one decimal place. **2**

d) Let $f(x) = \tan^{-1}\left(\frac{e^x - e^{-x}}{2}\right)$ and let $g(x) = \tan^{-1}(e^x)$.

(i) Show that $f'(x) = 2g'(x)$ **3**

(ii) Hence or otherwise, express $f(x)$ in terms of $g(x)$ **1**

End of Examination

<table>
<tr><th colspan="3">2022 Year 12 Mathematics Extension 1 Trial solutions</th></tr>
<tr><th>Q</th><th>Solution</th><th>Answer</th></tr>
<tr><td>1</td><td>

Option C:

$$x = 3 + 5\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{x-3}{5}$$

$$y = 4 + 5\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{y-4}{5}$$

$$\sin^2\theta + \cos^2\theta = \frac{(x-3)^2}{25} + \frac{(y-4)^2}{25}$$

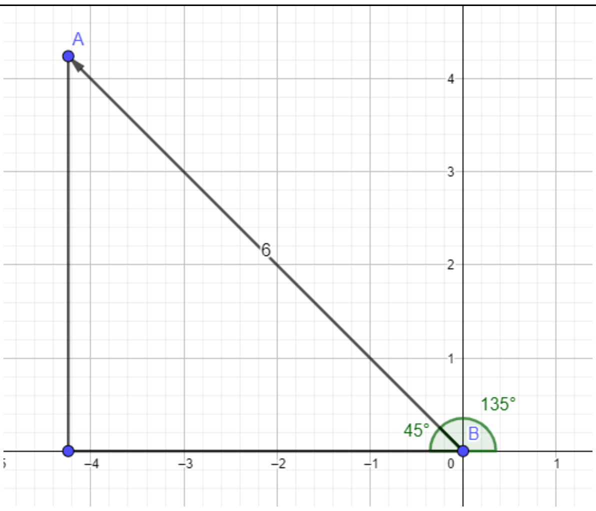
$$1 = \frac{(x-3)^2}{25} + \frac{(y-4)^2}{25}$$

$(x-3)^2 + (y-4)^2 = 25$

centre: $(3,4)$, radius: 5

The centre is $\sqrt{3^2+4^2} = 5$ units away from the origin, so the origin lies on the circumference.

</td><td>C</td></tr>
<tr><td>2</td><td>



$$\cos 45 = \frac{adj}{6}$$

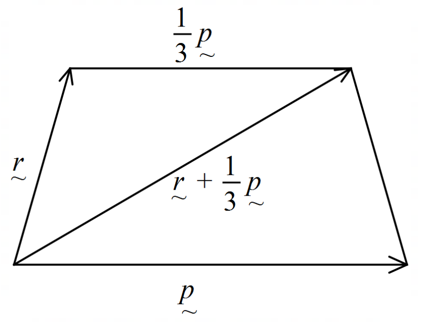
$$adj = 6\cos 45 = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$

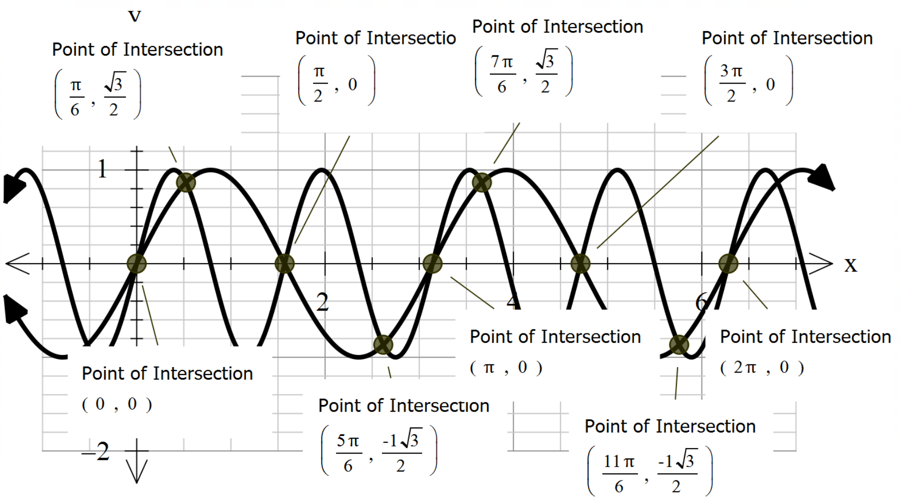
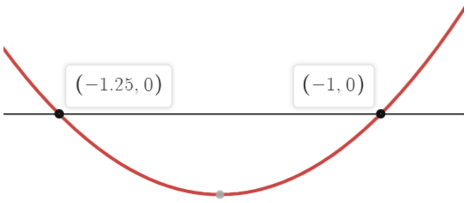
$$\sin 45 = \frac{opp}{6}$$

$$opp = 6\sin 45 = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$

$\therefore \quad \underset{\sim}{u} = -3\sqrt{2}\, i + 3\sqrt{2}\, j$

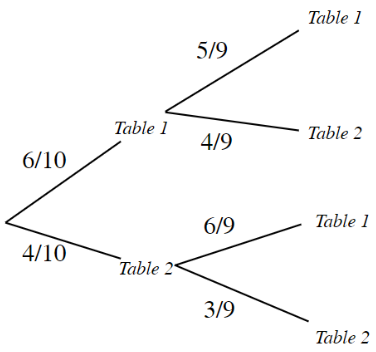
</td><td>B</td></tr>
<tr><td>3</td><td>The number of heads when a fair coin is tossed twice.</td><td>C</td></tr>
</table>

4	let $u = 2f(x)$ $\dfrac{du}{dx} = 2f\,'(x)$ $\dfrac{dy}{du} = -\dfrac{1}{\sqrt{1-u^2}}$ $\dfrac{dy}{dx} = -\dfrac{2f\,'(x)}{\sqrt{1-4[f(x)]^2}}$	D
5	 $\overrightarrow{PQ} = -\underset{\sim}{p} + \left(\underset{\sim}{r} + \frac{1}{3}\underset{\sim}{p}\right) = \underset{\sim}{r} - \frac{2}{3}\underset{\sim}{p}$	D
6	$y = e^x \Rightarrow \ln y = x$ $V = \pi\int x^2\,dy = \pi\int_1^{e^2} (\ln y)^2\,dy$	C
7	$\int \dfrac{1}{a^2+x^2}\,dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$ $\left[\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)\right]_0^k = \dfrac{\pi}{6}$ $\dfrac{1}{2}\tan^{-1}\left(\dfrac{k}{2}\right) - \dfrac{1}{2}\tan^{-1}0 = \dfrac{\pi}{6}$ $\tan^{-1}\left(\dfrac{k}{2}\right) = \dfrac{\pi}{3}$ $\dfrac{k}{2} = \tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$ $k = 2\sqrt{3}$	D

8	$\sin 4x = \sin 2x$ Solving graphically – 9 points of intersection 	
9	$\frac{dy}{dx} = \frac{y}{x^2}$ Denominator is always positive. Therefore gradient is positive where y>0 (Q1 and Q2) and negative where y<0 (Q3 and Q4)→ B	B
10	$np = 20 \times 0.8 = 16$ $nq = 20 \times 0.2 = 4$ As nq<5 the distribution is not approximately normal. As p=0.8 the peak will be to the right and the distribution is negatively skewed.	B
Q	**Solution**	**Marks**
11a	$\frac{x}{x+1} \leq 5$ $x \neq -1$ $x(x+1) \leq 5(x+1)^2$ $0 \leq 5(x+1)^2 - x(x+1)$ $0 \leq (x+1)(5x+5-x)$ $0 \leq (x+1)(4x+5)$ 	3 marks 1st mark for finding both critical values 2nd mark for using substitution to test regions or using graph of parabola 3rd mark for excluding x=-1 from solution

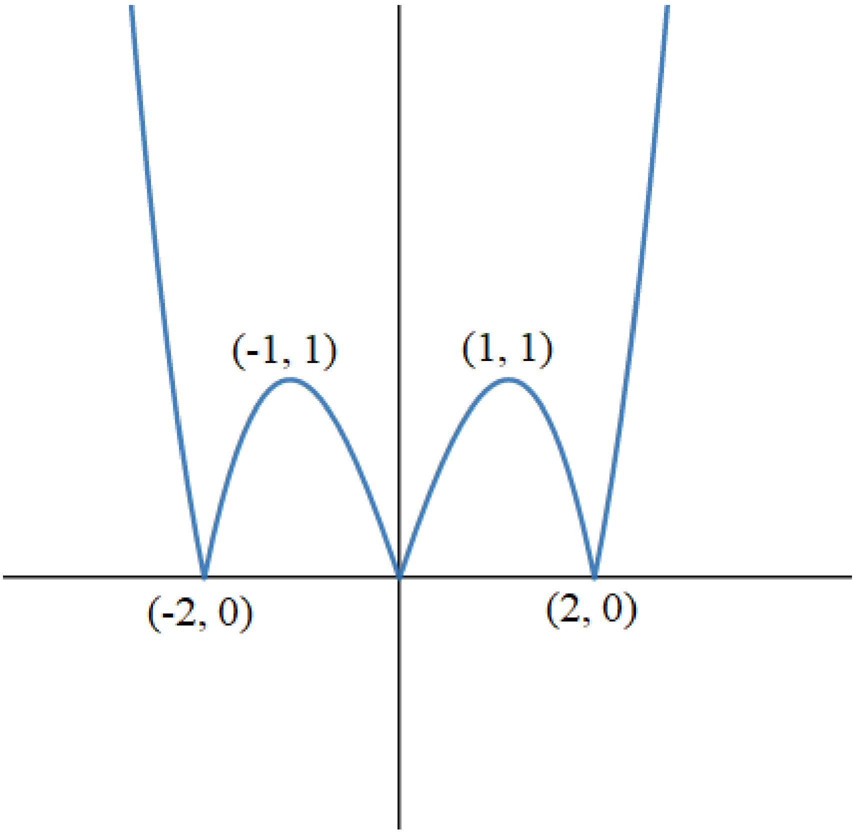
	$x \leq -\frac{5}{4}$ or $x \geq -1$ but $x \neq -1$ $\therefore x \leq -\frac{5}{4}$ or $x > -1$ **<u>Alternative method:</u>** $\frac{x}{x+1} \leq 5$ Consider the equation: $\frac{x}{x+1} = 5$ $x = 5x + 5$ $4x = -5$ $x = -\frac{5}{4}$ Critical values are -5/4 and -1. Substitute values in each region, e.g. When $x = -2$, $LHS = \frac{2}{2+1} = \frac{2}{3} \leq 5$ true When $x = -1.1$, $LHS = \frac{-1.1}{-1.1+1} = 11 \leq 5$ false When $x = 0$, $LHS = \frac{0}{0+1} = 0 \leq 5$ true $\therefore x \leq -\frac{5}{4}$ or $x > -1$	
11b	$\frac{508}{26} = 19.53\ldots$ Therefore a minimum of 20 rooms is needed	1 mark
11c	general term of $\left(3x^2 - \frac{1}{x}\right)^{12}$ $= {}^{12}\mathbf{C}_k (3x^2)^k (-x^{-1})^{12-k}$ $= {}^{12}\mathbf{C}_k\ 3^k\ x^{2k}\ (-1)^{12-k}\ x^{k-12}$ $= {}^{12}\mathbf{C}_k\ 3^k\ (-1)^{12-k}\ x^{3k-12}$ For independent term, $3k - 12 = 0 \Rightarrow k = 4$ $\Rightarrow {}^{12}\mathbf{C}_4\ 3^4\ (-1)^8 = 40095$	3 marks 1st mark for correct expression for the general term 2nd mark for determining correct value of *k*

		3rd mark for evaluating the constant term
11d	The polynomial can be written in the form: $P(x) = (x-3)(x-2)Q(x) + 2x + 1$ The remainder is P(3): $P(3) = 0 + 2(3) + 1$ $r = 7$ **Alternative method:** $P(x) = (x-3)(x-2)Q(x) + 2x + 1$ $\frac{P(x)}{x-3} = (x-2)Q(x) + \frac{2x+1}{x-3}$ $= (x-2)Q(x) + \frac{2x-6}{x-3} + \frac{7}{x-3}$ $= (x-2)Q(x) + 2 + \frac{7}{x-3}$ So the remainder is 7	2 marks 1st mark for expressing P(x) as (x-3)(x-2)*Q(x) + (2x+1) 2nd mark for substitution of $x = 3$
11e	$\sqrt{2}\cos\left(x+\frac{\pi}{4}\right) + \sqrt{2}\cos\left(x-\frac{\pi}{4}\right) = 1$ $\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right) + \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) = 1$ $\cos x - \sin x + \sin x + \cos x = 1$ $2\cos x = 1$ $\cos x = \frac{1}{2}$ $x = \pm\frac{\pi}{3}$ **Alternative method:** $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right) + \sqrt{2}\cos\left(x-\frac{\pi}{4}\right) = 1$ $\cos\left(x+\frac{\pi}{4}\right) + \cos\left(x-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ $2\cos x\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ (sum to product) $2\cos x \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ $\cos x = \frac{1}{2}$ $x = \pm\frac{\pi}{3}$	3 marks 1st mark for simplifying the LHS using trigonometric identities 2nd mark for obtaining a solution to the resulting equation 3rd mark for all solutions

11f	P(sitting at same table) $= P(\text{both at the 6-seater}) + P(\text{both at the 4-seater})$ $= \frac{6}{10} \times \frac{5}{9} + \frac{4}{10} \times \frac{3}{9}$ $= \frac{7}{15}$  **Alternative method:** total number of arrangements is = [no. of ways to split 10 people into 6 and 4] × [no. of ways to arrange 6 people in a circle] × [no. of ways to arrange 4 people in a circle] $= \binom{10}{6} \times 5! \times 3! = 151200$ Number of arrangements where the couple is seated at Table 1 (the 6-seater) is: [no. of ways to select 4 people to join them] × × [no. of ways to arrange 6 people in a circle] × [no. of ways to arrange 4 people in a circle] $= \binom{8}{4} \times 5! \times 3! = 50400$ Number of arrangements where the couple is seated at Table 2 (the 4-seater) is: [no. of ways to select 2 people to join them] × × [no. of ways to arrange 6 people in a circle] × [no. of ways to arrange 4 people in a circle] $= \binom{8}{2} \times 5! \times 3! = 20160$	3 marks for correct solution 2 marks for correctly finding the total number of arrangements or equivalent merit, and consideration of separate cases (both at Table 1 or both at Table 2) 1 mark for correctly arranging people in two circles (5!3!) or 1 mark for correct use of nCk to select a group of 6 or 4, or equivalent merit

	Therefore, $P(couple\ at\ same\ table) = \frac{50400+20160}{151200} = \frac{70560}{151200} = \frac{7}{15}$ **<u>Alternative method:</u>** The 10 people are split up into a group of 6 and 4. Total no. of combinations = $\binom{10}{6} = 210$ No. of combinations where the couple is in the group of 6 = no. of ways to choose 4 people to join them = $\binom{8}{4} = 70$ No. of combinations where the couple is in the group of 4 = no. of ways to choose 2 people to join them = $\binom{8}{2} = 28$ $P(couple\ at\ same\ table) = \frac{70+28}{210} = \frac{7}{15}$	
12a	Test n = 1 LHS = 2 RHS = 2 therefore true for n = 1 assume n = k is true $2 + 4 + \cdots + 2k = k(k-1) + 2 \qquad k \geq 1$ Test if it is true for n = k + 1 $2 + 4 + \cdots + 2k + 2(k+1) = k(k+1) + 2$ $LHS = k(k-1) + 2 + 2(k+1)$ $= k^2 + k + 4$ $RHS = k^2 + k + 2$ $LHS \neq RHS$ hence the claim is not true for n = k+ 1 so fails at the inductive step. Students who showed that the statement failed for n = 2 or n = 3 were awarded the marks. However, this is not how the question was intended.	1 mark for testing n = 1 1 mark for proving LHS does not equal RHS and explaining it fails at this step. NOTE: Must have the correct working. Students who did not have the correct working were penalised.

12b	$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{4-4x^2}}\,dx$ $= \int_0^{\frac{1}{2}} \frac{1}{\sqrt{4(1-x^2)}}\,dx$ $= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ $= \frac{1}{2}\sin^{-1}\frac{1}{2} - \frac{1}{2}\sin^{-1} 0$ $= \frac{1}{2} \times \frac{\pi}{6}$ $= \frac{\pi}{12}$	2 marks for correct working and solution. 1 mark for factoring out the 4 and multiplying the integral by ½.
12c	Method 1 $\underset{\sim}{\mathbf{p}} = \lambda\underset{\sim}{\mathbf{q}}$ $-6\underset{\sim}{\mathbf{i}} + a\underset{\sim}{\mathbf{j}} = \lambda(a+1)\underset{\sim}{\mathbf{i}} - 7\lambda\underset{\sim}{\mathbf{j}}$ $a = -7\lambda$ $\lambda = -\frac{a}{7}$ $-6 = \lambda(a+1)$ $-6 = -\frac{a}{7}(a+1)$ $42 = a^2 + a$ $a^2 + a - 42 = 0$ $(a+7)(a-6) = 0$ $a = -7$ or $a = 6$	1 mark for correct working 1 mark for correct solutions of a

	Method 2 gradients must be equal $\frac{a+1}{-7} = -\frac{6}{a}$ $a^2 + a - 42 = 0$ $(a+7)(a-6) = 0$ $a = -7$ or $a = 6$	
12d		3 marks for correct solution. 2 marks for applying both transformations 1 mark for locating coordinates

12e i	standard deviation = $\sqrt{50 \times 0.3 \times 0.7}$ $= 3.24$	1 mark
12 e ii	mean = 50×0.3 $= 15$ $P(X = 15) = {}^{50}\mathbf{C}_{15} \times 0.3^{15} \times 0.7^{35}$ $= 0.12$	1 mark for the mean 1 mark for the final answer
12f	$\frac{dy}{dx} = y^3$ $y^{-3} dy = 1\ dx$ $\frac{1}{-2y^2} = x + C$ $-\frac{1}{2} = 0 + C$ $c = -\frac{1}{2}$ $\frac{1}{y^2} = \frac{1}{1-2x}$ $y^2 = \frac{1}{1-2x}$ $y = \frac{1}{\sqrt{1-2x}}$ since $x = 0$ $y = 1$	3 marks for correct working and solution. 2 marks for correct C value and correct expression for y squared. 1 mark for correct C value
13ai	$\frac{2}{3} \times 0.6 + \frac{1}{3} \times 0.7 = \frac{19}{30}$	1 mark for correct solution
13a(ii)	$P(X \leq 8) = 1 - P(X = 9) - P(X = 10)$ $= 1 - {}^{10}\mathbf{C}_9\left(\frac{19}{30}\right)^9\left(\frac{11}{30}\right)^1 - {}^{10}\mathbf{C}_{10}\left(\frac{19}{30}\right)^{10}\left(\frac{11}{30}\right)^0$ $= 0.93$	2 marks for correct solution 1 mark for writing expression for $P(X \leq 8)$
13a(iii)	Method 1: Let $\hat{p}$ be the random variable representing the sample proportion of determinate tomatoes. $E(\hat{p}) = p = \frac{19}{30}$ Standard deviation $(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{\frac{19}{30}\times\frac{11}{30}}{33}} = \sqrt{\frac{19}{2700}}$	3 marks for correct solution 2 marks for z score 1 mark for std dev

	$P\left(\hat{p} \geq 0.55\right) = P\left(z \geq \dfrac{0.55 - \frac{19}{30}}{\sqrt{\frac{19}{2700}}} \right) = P(z \geq -0.993)$ $= 1 - 0.1611$ (from table) ≈ 0.8389 Method 2: $E(X) = np = 33 \times \frac{19}{30} = 20.9$ $\sigma(X) = \sqrt{33 \times \frac{19}{30} \times \frac{11}{30}} \approx 2.768$ $33 \times 55\% = 18.15$ $P(X \geq 18.15) = P\left(z \geq \frac{18.15 - 20.9}{2.768}\right) = P(z \geq -0.99)$ continues the same as in Method 1 *Marker's comments:* • *Some students mixed values required to calculate z score (e.g. used standard deviation from method 1 but mean from method 2)*	
13b	**Auxiliary Angle method:** $r = \sqrt{2}$ $\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$ $\sin\theta + \cos\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$ $\therefore \sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$ $\theta + \frac{\pi}{4} = \pi - \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right), 2\pi + \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right), \frac{\pi}{4} \leq \theta + \frac{\pi}{4} \leq 2\pi + \frac{\pi}{4}$ $\theta + \frac{\pi}{4} = 2.7802\ldots, 6.6445\ldots$ $\theta = 1.995, 5.859$ $\sin\theta + \cos\theta = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$ would also work for this method **t formulae method:**	3 marks for correct solution 2 marks for writing in auxiliary angle form 2 marks for correct values of t 1 mark for finding r or α 1 mark for obtaining correct t expression 1 mark for sin2th=-3/4

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$\sin\theta + \cos\theta = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \frac{1+2t-t^2}{1+t^2}$

$\frac{1+2t-t^2}{1+t^2} = \frac{1}{2}$

$3t^2 - 4t - 1 = 0$

$t = \frac{4 \pm \sqrt{28}}{6} = \frac{2 \pm \sqrt{7}}{3}$

$\frac{\theta}{2} = \tan^{-1}\left(\frac{2+\sqrt{7}}{3}\right) \Rightarrow \theta = 1.995$

$\frac{\theta}{2} = \pi - \tan^{-1}\left|\frac{2-\sqrt{7}}{3}\right| \Rightarrow \theta = 5.859$

Check that $\theta = \pi$ is not a solution:

$\sin\pi + \cos\pi = 0 + (-1) = -1 \neq \frac{1}{2}$

If using t-formulae method, must check that pi is not a solution. In this case, marks were not attached to this step

Alternative method:

$(\sin\theta + \cos\theta)^2 = \frac{1}{4}$

$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = \frac{1}{4}$

$\sin 2\theta + 1 = \frac{1}{4}$

$\sin 2\theta = -\frac{3}{4}$

$2\theta = \pi + \sin^{-1}\left(\frac{3}{4}\right), 2\pi - \sin^{-1}\left(\frac{3}{4}\right), 3\pi + \sin^{-1}\left(\frac{3}{4}\right), 4\pi - \sin^{-1}\left(\frac{3}{4}\right)$

$\theta = 1.995, 2.718, 5.136, 5.859$

Check and eliminate

$\therefore \quad \theta = 1.995, 5.859$

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<tr><td>13c</td><td>

$|\underset{\sim}{u}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$|\underset{\sim}{v}| = 3$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|\,|\underset{\sim}{v}| \cos\theta$

$0 + 3 = 3\sqrt{5}\cos\theta$

$\cos\theta = \frac{1}{\sqrt{5}} \Rightarrow \sin\theta = \frac{2}{\sqrt{5}}$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}} = \frac{4}{5}$

Marker's comments:

- *Show question, therefore all steps need to be shown. Solutions such as one below, were not awarded full marks*

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3 marks for correct solution

2 marks for substantial progress and obtaining cosθ

1 mark for magnitude of vectors

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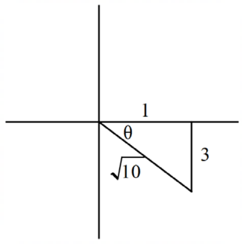
	$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$ $\sin 2\theta = \sin\left(2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) = \frac{4}{5}$	
13d	$y^2 = (2\cos x - 1)^2 = 4\cos^2 x - 4\cos x + 1$ $V = \pi\int y^2\,dx$ $= \pi\int_0^{\frac{\pi}{3}} 4\cos^2 x - 4\cos x + 1\,dx$ $= \pi\int_0^{\frac{\pi}{3}} 4 \times \frac{1}{2}(1 + \cos 2x) - 4\cos x + 1\,dx$ $= \pi\int_0^{\frac{\pi}{3}} 3 + 2\cos 2x - 4\cos x\,dx$ $= \pi\,[3x + \sin 2x - 4\sin x]_0^{\frac{\pi}{3}}$ $= \pi\left[\pi + \sin\left(\frac{2\pi}{3}\right) - 4\sin\left(\frac{\pi}{3}\right) - (0 + 0 - 0)\right]$ $= \pi\left[\pi + \frac{\sqrt{3}}{2} - \frac{4\sqrt{3}}{2}\right]$ $= \pi\left[\pi - \frac{3\sqrt{3}}{2}\right]$	3 marks for correct solution 2 marks for 1 integration error only and correct substitution 2 marks for correct integration of y^2 1 mark for correct expression for y^2
14a	Base case: Let n=1 $8^n - 5^n = 8^1 - 5^1 = 3$ which is a multiple of 3 $S(k)$: Let it be true for $n = k$ i.e. $8^k - 5^k = 3M$, k, $M \in \mathbb{Z}^+$ (positive integers) $8^k = 3M + 5^k$ * $S(k+1)$: show true for $n = k+1$ RTP $8^{k+1} - 5^{k+1} = 3Q$ $(Q \in \mathbb{Z}^+)$ LHS = $8^k \times 8^1 - 5^k \times 5^1$ $= (3M + 5^k) \times 8 - 5 \times 5^k$ from * $= 24M + 3 \times 5^k$ $= 3(8M + 5^k)$ which is divisible by 3 as $M, k \in \mathbb{Z}^+$	3 marks – correct solution 2 marks – uses inductive step 1 mark – proves true for base case

14b	Let $\tan^{-1}(-3) = \theta$ $\tan\theta = -3$ tan is negative in Q2 and Q4 however the range of $\tan^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (Q 4 and Q 1) so θ is in Q4 (where $\cos\theta > 0$) so $\cos\theta = \frac{1}{\sqrt{10}}$ hence $\cos(\tan^{-1}(-3)) = \frac{1}{\sqrt{10}}$  *Markers Note – Many students answered* $\pm\frac{1}{\sqrt{10}}$ *achieving one mark only as the negative answer is outside the range of* $\tan^{-1}x$	2 marks – correct solution 1 mark – Identifies correct quadrant, or gives q2 answer.
14ci	$\frac{dm}{dt}$ = mass rate of salt pumped into the tank - mass rate of salt pumped out of the tank = concentration pumped in (in kg/L) × volume rate pumped in (in L/min) - concentration pumped out (in kg/L) × volume rate pumped out (in L/min) $= 0 \times 20 - \frac{m}{V_{tank}} \times 15$ $= \frac{-15m}{V_{tank}}$ $= \frac{-15m}{200+5t}$ $= \frac{-3m}{40+t}$	1 mark -clearly demonstrated
14cii	$\frac{dm}{dt} = -\frac{3m}{40+t}$ $\int\frac{dm}{m} = -3\int\frac{dt}{40+t}$ $\ln\lvert m\rvert = -3\ln\lvert 40+t\rvert + c$ ln $+3\ln(40+t) = c$ can remove || as m and t are >0) $\ln m + \ln(40+t)^3 = c$ $\ln(m(40+t)^3) = c$ $m(40+t)^3 = e^c$ $m = \frac{A}{(40+t)^3}$ where $A = e^c$ *when* $t = 0$ $m = 50$ $50 = \frac{A}{40^3}$ $A = 50 \times 40^3 = 3200000$ $m = \frac{3200000}{(40+t)^3}$	3 marks – correct solution fully detailed 2 marks – correct integration and some correct simplification 1 mark – separates integral and integrates correctly

14ciii	Volume is increasing by 5L/min. Needs to increase by 300L to overflow. This will take 60 minutes. When t=60: $m = \dfrac{3200000}{(40+60)^3}$ $= 3.2\text{kg}$	2 marks – correct solution 1 mark – finds t=60
14di	$f(x) = \tan^{-1}\left(\dfrac{e^x - e^{-x}}{2}\right)$ $f'(x) = \dfrac{\frac{1}{2}(e^x + e^{-x})}{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}$ $= \dfrac{\frac{1}{2}(e^x + e^{-x})}{1 + \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right)}$ $= \dfrac{2(e^x + e^{-x})}{4 + ((e^{2x} - 2 + e^{-2x}))}$ $= \dfrac{2(e^x + e^{-x})}{2 + e^{2x} + e^{-2x}})$ $= \dfrac{2\left(e^x + \frac{1}{e^x}\right)}{2 + e^{2x} + \frac{1}{e^{2x}}}$ $= \dfrac{2(e^{3x} + e^x)}{2e^{2x} + e^{4x} + 1})$ $= \dfrac{2e^x(e^{2x} + 1)}{(e^{2x} + 1)^2}$ $= \dfrac{2e^x}{e^{2x} + 1}$ $= 2g'(x)$ $\therefore f'(x) = 2g'(x)$ as required	3 marks – correct solution including <u>showing</u> factorising and cancelling to achieve f'(x)=2g'(x) 2 marks – expands the square on the denominator correctly and some correct tidying up 1 mark – correctly differentiates f(x) (no marks assigned for finding g'(x))

14dii	$f(x) = \int f'(x)\, dx$ $= 2\int g'(x)\, dx$ (from (i)) $= 2g(x) + c$ $= 2\tan^{-1}(e^x) + c$ when $x = 0$ $f(x) = \tan^{-1}\left(\frac{e^0 - e^0}{2}\right)$ $f(0) = \tan^{-1}(0) = 0$ $\therefore 2\tan^{-1}(e^0) + c = 0$ $2\tan^{-1}(1) + c = 0$ $2 \times \frac{\pi}{4} + c = 0$ $c = -\frac{\pi}{2}$ and $f(x) = 2g(x) - \frac{\pi}{2}$	1 mark – correct solution including using initial conditions to correctly find the value of c